On Strategic Complementarities in Discontinuous Games with Totally Ordered Strategies

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Abstract This paper studies the existence of a pure strategy Nash equilibrium in games with strategic complementarities where the strategy sets are totally ordered. By relaxing the conventional conditions related to upper semicontinuity and single crossing, we enlarge the class of games to which monotone techniques are applicable. The results are illustrated with a number of economics-related examples.

Keywords Discontinuous game; Strategic complementarities; Better-reply security; Directional transfer single crossing; Increasing correspondence

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1 Introduction

Upper semicontinuity, quasisupermodularity, and Milgrom and Shannon's (1994) single crossing are sufficient for a normal-form game where the strategy sets are compact lattices in Euclidean spaces to have a pure strategy Nash equilibrium.[2] In many economics-related games, the strategy sets are totally ordered. In such cases, both upper semicontinuity and single crossing are excessively demanding. The focus of this paper is on relaxing both of the two conditions.

In games with strategic complementarities, the best-reply correspondences are, usually, assumed to be nonempty-valued and subcomplete-sublattice-valued. These propitious properties of the best-reply correspondences are achieved by making a not entirely innocuous assumption, namely that each payoff function is upper semicontinuous in own strategy, which noticeably narrows the class of games in which equilibrium existence can be studied with the aid of lattice-theoretic tools. In this paper, upper semicontinuity is replaced with one of the following pairs of conditions: either with Tian and Zhou's (1995) transfer weak upper continuity and directional upper semicontinuity or with Reny's (1999) better-reply security and directional upper semicontinuity, thereby making it possible to cover new classes of games to which the seminal contributions by Vives (1990), Milgrom and Shannon (1994), and Reny (1999) cannot be applied.

For games where the payoff functions are transfer weakly upper continuous in own strategies, single crossing is generalized to directional transfer single crossing. The word 'directional' means that single crossing is divided into upward single crossing and downward single crossing, and the word 'transfer' reflects the fact that, in this paper, the notion of an increasing correspondences is understood in Smithson's (1971) and Fujimoto's (1984) sense. We illustrate the interplay of the different notions with the aid of a partnership game (Example 4) and a war of attrition game (Example 5). In the latter, player 1's payoff function satisfies upward single crossing and player 2's payoff function satisfies downward single crossing when the natural order relation on player 2's strategy set is reversed.

The lattice-theoretic approach covers a large number of oligopoly models (see, e.g., Roberts and Sonnenschein, 1976; Vives, 1990; Amir, 1996; Vives, 1999; and Amir and De Castro, 2015). However, the classic Bertrand oligopoly model with homogeneous products is not one of them since its payoff functions are too discontinuous. At the same time, in the two-firm case, for example, if, initially, the profit-maximizing firms charge prices exceeding the unit cost of production, then any of them has no incentive to lower its price in reaction to an increase in the price charged by its rival. On the other hand, if the demand curve has a conventional convex shape, the quasiconcavity of the Bertrand duopoly game tends to fail, and,

[2] See, for general reviews, Vives (1999), Amir (2005), and Vives (2005).

consequently, it might be impossible to apply Reny's (1999) equilibrium existence theorem and any of its generalizations.[3]

We handle the equilibrium existence problem in games where the best-reply correspondences are not necessarily nonempty-valued everywhere in two steps. The first step employs lattice-theoretic tools and directional upper semicontinuity to investigate the existence of ε-equilibria, and the second step relies on the fact that, in the better-reply secure games, the cluster points of a sequence of ε-Nash equilibria are Nash equilibria. In order to express strategic complementarities in terms of ε-best-reply correspondences, two more directional modifications of single crossing are introduced. The proposed equilibrium existence conditions are illustrated on a nonquasiconcave Bertrand duopoly model with homogeneous products (Example 6).

The structure of the paper is as follows. Section 2 contains some theoretical underpinnings necessary for studying strategic complementarities in discontinuous games. The main results of the paper are presented in Section 3, and the illustrating examples are provided in Section 4.

2 Preliminaries

This section provides some lattice-theoretic and topological definitions and auxiliary results.

2.1 Posets

Given a nonempty set P, a binary relation $\preceq$ on P is a partial order if it is reflexive, antisymmetric, and transitive. The pair $(P, \preceq)$ is a partially ordered set or poset, though it is often said that P is a poset if there is no ambiguity regarding the order relation involved. A poset P is totally ordered if every $x, y \in P$ are comparable, that is, $x \preceq y$ or $y \preceq x$. Denote the asymmetric part of the relation $\preceq$ by $\prec$.

Let $(P, \preceq)$ be a poset. An element $m \in P$ is a maximal element (resp., a minimal element) of P if for all $p \in P$, $m \preceq p$ (resp., $p \preceq m$) implies $m = p$. An element $m \in P$ is the greatest element (resp., the least element) of P if $p \preceq m$ (resp., $m \preceq p$) for all $p \in P$. Let $S \subset P$. An upper (resp., lower) bound for S is an element $p \in P$ such that $s \preceq p$ (resp., $p \preceq s$) for all $s \in S$. The least upper bound (resp., the greatest lower bound) of S is also called the join (resp., the meet) of S and is denoted by $\bigvee S$ (resp., $\bigwedge S$). The set P is a lattice if every pair of elements of P has a meet and a join. It is a complete lattice if P has arbitrary meets and arbitrary joins. A nonempty subset S is a chain in P if S is totally ordered by $\preceq$.

[3] See McLennan, Monteiro and Tourky (2011), Barelli and Meneghel (2013), Carmona and Podczeck (2016), and Reny (2016).

The interval topology on a totally ordered set P is the topology generated by the closed subbase consisting of the sets $[a, +\infty) = \{p \in P : a \preceq p\}$ and $(-\infty, a] = \{p \in P : p \preceq a\}$ where $a \in P$.[4] Every totally ordered set in its interval topology is a normal Hausdorff space. A chain's compactness in the interval topology is equivalent to its completeness (see, e.g., Birkhoff, 1967, p. 241-242).

By an ordered topological space we mean a nonempty set P equipped with a partial order $\preceq$ and a topology such that the intervals $[a, +\infty)$ and $(-\infty, a]$ are closed for each $a \in P$. It is useful to notice that every totally ordered compact space is a complete chain, since the compactness of the topological space implies its compactness in the interval topology.

2.2 Upper semicontinuity

This subsection gives some basic facts about upper semicontinuous functions. First, we introduce two types of directional upper semicontinuity.

Definition 1 *Let P be a totally ordered compact space. A function $f : P \to \mathbb{R}$ is upward (resp., downward) upper semicontinuous if for every increasing (resp., decreasing) net $\{p_\alpha\}$ of elements of P, $\limsup_\alpha f(p_\alpha) \leq f(\bigvee_\alpha p_\alpha\})$ (resp., $\limsup_\alpha f(p_\alpha) \leq f(\bigwedge_\alpha p_\alpha)$). A function $f : P \to \mathbb{R}$ is order upper semicontinuous if it is upward and downward upper semicontinuous.*

The definition of a real-valued, order upper semicontinuous function defined on a complete lattice can be found in Milgrom and Roberts (1990).

Each of the directional upper semicontinuity properties is weaker than upper semicontinuity for functions defined on a totally ordered compact space.

Example 1 Consider the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1$ if $x \in [0, 1)$ and $f(1) = 0$. This function is lower semicontinuous and downward upper semicontinuous under the natural order on $[0, 1]$.

Now let us look at some generalizations of upper semicontinuity from a topological point of view. Let P be a topological space. A function $f : P \to \mathbb{R}$ is upper semicontinuous at p if for any $\lambda \in \mathbb{R}$ such that $f(p) < \lambda$, there exists a neighborhood $\mathcal{N}(p)$ of p such that $f(s) < \lambda$ for all $s \in \mathcal{N}(p)$. A function $f : P \to \mathbb{R}$ is upper semicontinuous if it is upper semicontinuous at every $p \in P$. Another equivalent definition of upper semicontinuity is the following: A function

[4] Another approach to introducing a topology on a complete lattice focuses on order convergence. On any complete chain, the order topology coincides with the interval topology (see, e.g., Birkhoff, 1967, p. 244).

$f : P \to \mathbb{R}$ is upper semicontinuous at $p \in P$ if and only if $p_\alpha \to p$ in P implies that $\limsup_\alpha f(p_\alpha) \le f(p)$. This definition is used to show the following lemma.

Lemma 1 *Let P be a totally ordered compact space. A function $f : P \to \mathbb{R}$ is upper semicontinuous if and only if it is order upper semicontinuous.*

The proof of Lemma 1 is provided in the Appendix for the sake of the reader's convenience.

The notion of an upper semicontinuous function was relaxed by Campbell and Walker (1990) and Tian and Zhou (1995). Let P be a topological space. A function $f : P \to \mathbb{R}$ is upper continuous if for any points $p,\ s \in P$, $f(p) < f(s)$ implies that there exists a neighborhood $\mathcal{N}(p)$ of p such that $f(r) < f(s)$ for all $r \in \mathcal{N}(p)$. It is another equivalent definition of an upper semicontinuous function. Replacing the latter inequality in the definition of an upper continuous function with its weak counterpart leads to a generalization of the notion of an upper semicontinuous function. A function $f : P \to \mathbb{R}$ is weakly upper continuous if for any points $p,\ s \in P$, $f(p) < f(s)$ implies that there exists a neighborhood $\mathcal{N}(p)$ of p such that $f(r) \le f(s)$ for all $r \in \mathcal{N}(p)$ (see Campbell and Walker, 1990). The set of maximum points of a weakly upper continuous function on a compact set is nonempty but not necessarily closed. An important generalization of the notion of a weakly upper continuous function is that of a transfer weakly upper continuous function, due to Tian and Zhou (1995). A function $f : P \to \mathbb{R}$ is transfer weakly upper continuous if for any points $p,\ s \in P$, $f(p) < f(s)$ implies that there exist $u \in P$ and a neighborhood $\mathcal{N}(p)$ of p such that $f(r) \le f(u)$ for all $r \in \mathcal{N}(p)$. A necessary and sufficient condition for a function defined on a compact subset of P to attain its maximum on the set is the transfer weak upper continuity of the function (see Tian and Zhou, 1995, Theorem 1).

2.3 Increasing correspondences

Let P and T be posets. A function $f : P \to T$ is increasing if $p \preceq s$ in P implies $f(p) \preceq f(s)$ in T. Since, according to Tarski's fixed point theorem (Tarski, 1955), every increasing function from a complete lattice to itself has a fixed point, the problem of existence of a fixed point for an increasing correspondence is often reduced to showing that it has a single-valued increasing selection. However, depending on needs, several definitions of an increasing correspondence can be employed.

Definition 2 *Let P and T be posets. A nonempty-valued correspondence $F : P \twoheadrightarrow T$ is increasing upward (resp., downward) if $p \preceq s$ in P and $u \in F(p)$ (resp., $v \in F(s)$) imply that there exists $v \in F(s)$ (resp., $u \in F(p)$) such that $u \preceq v$.*

If a correspondence $F : P \twoheadrightarrow T$ is increasing upward and downward, it is called increasing.

Smithson (1971) and Fujimoto (1984) extended Tarski's fixed point theorem to increasing correspondences (see, for more up-to-date results, Heikkila and Reffett, 2006; Carl and Heikkila, 2011).

In economics literature, a stronger notion of an increasing correspondence is more popular than the one given just above.

Definition 3 *A nonempty-valued correspondence $F : P \twoheadrightarrow T$ is Veinott-increasing upward (resp., downward) if $p \preceq s$ in P, $u \in F(p)$ and $v \in F(s)$ imply that $u \vee v \in F(s)$ (resp., $u \wedge v \in F(p)$). If a correspondence $F : P \twoheadrightarrow T$ is Veinott-increasing upward and downward, it is called Veinott-increasing.*

Another, more traditional name for a Veinott-increasing correspondence is a correspondence increasing in the induced (strong) set order (see, e.g., Topkis, 1998, p. 32). It is easy to see that: (1) the notion of an increasing correspondence is considerably less demanding than the notion of a Veinott-increasing correspondence; (2) an increasing correspondence need not have an increasing single-valued selection.

Example 2 Consider the correspondence $F : [0, 1] \twoheadrightarrow [0, 1]$ defined by $F(p) = [\frac{1}{3}p, \frac{1}{3}p + \frac{1}{3}] \backslash \{p\}$. The set $[0, 1]$, equipped with the conventional order $\leq$, is a complete chain. It is clear that F is an increasing correspondence with no fixed points. At the same time, F is not Veinott-increasing. For example, $\frac{5}{12} \in F(\frac{1}{4})$, $\frac{1}{4} \in F(\frac{1}{2})$, and $\frac{5}{12} \wedge \frac{1}{4} = \frac{1}{4} \notin F(\frac{1}{4})$.

The next result is straightforward, but helpful.

Lemma 2 *Let P and T be posets, and let $F : P \twoheadrightarrow T$ be an increasing upward (resp., downward) correspondence with nonempty values. If $\bigvee F(p) \in F(p)$ (resp., $\bigwedge F(p) \in F(p)$) for every $p \in P$, then F has an increasing selection.*

Proof. In order to show the claim, it suffices to verify that the function $f : P \to T$ defined by $f(p) = \bigvee F(p)$ (resp., $f(p) = \bigwedge F(p)$) for $p \in P$ is increasing.

If, for example, the correspondence F is increasing upward and $p_1, p_2 \in P$ such that $p_1 \preceq p_2$, then there exists $u \in F(p_2)$ such that $f(p_1) \preceq u$. Since $u \preceq f(p_2)$, we have that $f(p_1) \preceq f(p_2)$. ■

2.4 Directional transfer single crossing

The single-crossing property generalizes the property of increasing differences and has found numerous applications in economics (see, e.g., Edlin and Shannon, 1998;

Athey, 2001; Reny and Zamir, 2004; Quah and Strulovici, 2009, 2012; and Reny, 2011). This section contains several generalizations of the single-crossing property.

Let P and T be posets and let $f : P \times T \to \mathbb{R}$. The function f has increasing differences in (p, t) if for all $p'' \succ p'$, $f(p'', t) - f(p', t)$ is increasing in t. The function f satisfies the single-crossing property in $(p; t)$ if for all $p'' \succ p'$ and $t'' \succ t'$, $f(p'', t') - f(p', t') > 0$ implies that $f(p'', t'') - f(p', t'') > 0$ and $f(p'', t') - f(p', t') \geq 0$ implies that $f(p'', t'') - f(p', t'') \geq 0$. The function f satisfies the weak single-crossing property in $(p; t)$ if for all $p'' \succ p'$ and $t'' \succ t'$, $f(p'', t') - f(p', t') > 0$ implies that $f(p'', t'') - f(p', t'') \geq 0$.

The single-crossing property is usually used along with the quasisupermodularity property. Let P be a lattice. A function $f : P \to \mathbb{R}$ is quasisupemodular if for all p' and p'' in P, $f(p' \wedge p'') \leq f(p')$ implies that $f(p'') \leq f(p' \vee p'')$ and $f(p' \wedge p'') < f(p')$ implies that $f(p'') < f(p' \vee p'')$. Clearly, every real-valued function defined on a totally ordered set is quasisupermodular.

The following lemma is a corollary of Theorem 4 of Milgrom and Shannon (1994).

Lemma 3 *Let P be a lattice, T be a poset, and let $f : P \times T \to \mathbb{R}$. Let the correspondence $M : T \twoheadrightarrow P$ defined by $M(t) = \{p \in P : f(p, t) = \sup_{z \in P} f(z, t)\}$ be nonempty-valued. Then it is Veinott-increasing if f is quasisupermodular in p and satisfies the single-crossing property in $(p; t)$.*

Since, in discontinuous games, best-reply correspondences are often not Veinott-increasing, we need to introduce directional transfer single crossing.

Definition 4 *Let P and T be posets and let $f : P \times T \to \mathbb{R}$. The function f satisfies the upward (resp., downward) transfer single-crossing property in $(p; t)$ if for all $p' \prec p''$ (resp., $p' \succ p''$) and $t' \prec t''$(resp., $t' \succ t''$), $f(p'', t') - f(p', t') \geq 0$ implies that $f(\widehat{p}, t'') - f(p', t'') \geq 0$ for some $\widehat{p} \in P$ with $\widehat{p} \succeq p''$ (resp., $\widehat{p} \preceq p''$).*

If, in Definition 4, $\widehat{p} = p''$, then the word 'transfer' can be omitted. Obviously, every function $f : P \times T \to \mathbb{R}$ satisfying the upward (resp., downward) single-crossing property in $(p; t)$ also satisfies the upward (resp., downward) transfer single-crossing property in $(p; t)$.

The upward and downward single-crossing properties are the two sides of Milgrom and Shannon's (1994) single-crossing property (see also Milgrom, 2004, ch. 4). For the first-price sealed-bid auctions with incomplete information, a similar reformulation of Athey's (2001) single-crossing condition can be found in Reny and Zamir (2004).

Lemma 4 *Let P and T be posets and let $f : P \times T \to \mathbb{R}$. The function f satisfies the single-crossing property in $(p; t)$ if and only if it satisfies the upward and downward single-crossing properties in $(p; t)$.*

Proof. Assume that f satisfies the single-crossing property in $(p;t)$. We only need to show that it has the downward single-crossing property in $(p;t)$. Let $p' \succ p''$ in P, $t' \succ t''$ in T, and $f(p'',t') - f(p',t') \geq 0$. Assume, by contradiction, that $f(p'',t'') - f(p',t'') < 0$. Then, by single crossing, $f(p',t') - f(p'',t') > 0$, a contradiction.

Now assume that f has the upward and downward single-crossing properties in $(p;t)$. Let $p'' \succ p'$ in P, $t'' \succ t'$ in T, and $f(p'',t') - f(p',t') > 0$. We need to show that $f(p'',t'') - f(p',t'') > 0$. Assume, by contradiction, that $f(p',t'') - f(p'',t'') \geq 0$. Then, by downward single crossing, $f(p',t') - f(p'',t') \geq 0$, a contradiction. ■

The next lemma explains why the directional transfer single-crossing properties are useful in game-theoretic applications.

Lemma 5 *Let P be a totally ordered set and T be a poset. Let $f : P \times T \to \mathbb{R}$ satisfy the upward (resp., downward) transfer single-crossing property in $(p;t)$. If the correspondence $M : T \twoheadrightarrow P$ defined by $M(t) = \{p \in P : f(p,t) = \sup_{z \in P} f(z,t)\}$ is nonempty-valued, then it is increasing upward (resp., downward).*

Proof. Assume, for example, that f satisfies the upward transfer single-crossing property in $(p;t)$. Pick some t' and t'' in T with $t'' \succ t'$. Pick some $p' \in M(t')$. We need to show that there exists $p'' \in M(t'')$ such that $p'' \succeq p'$. By way of contradiction, assume that it is not the case; that is, $p'' \prec p'$ for every $p'' \in M(t'')$. Fix some $p'' \in M(t'')$. Then, by the upward single-crossing property, $f(p',t') - f(p'',t') \geq 0$ implies that $f(\widehat{p},t'') - f(p'',t'') \geq 0$ for some $\widehat{p} \in P$ with $\widehat{p} \succeq p'$; that is, $\widehat{p} \in M(t'')$, a contradiction. ■

A statement, similar to Lemma 5, for Veinott-increasing upward (downward) correspondences is the following.

Lemma 6 *Let P be a totally ordered set and T be a poset. Let $f : P \times T \to \mathbb{R}$ satisfy the upward (resp., downward) single-crossing property in $(p;t)$. If the correspondence $M : T \twoheadrightarrow P$ defined by $M(t) = \{p \in P : f(p,t) = \sup_{z \in P} f(z,t)\}$ is nonempty-valued, then it is Veinott-increasing upward (resp., downward).*

It is useful to notice that upward single crossing in Lemma 6 can not be relaxed to Shannon's (1995) weak single crossing.

Example 3 Let $P = T = \{0,1\}$ and $f(p,t) = \max\{1-p, 1-t\}$ for all $(p,t) \in P \times T$. Then the function satisfies the weak single-crossing property in $(p;t)$ trivially because $f(1,0) - f(0,0) = 0$. However, $M(0) = \{1,0\}$ and $M(1) = \{0\}$.

In order to be able to handle games where the best-reply correspondences are not necessarily nonempty-valued everywhere, we now introduce approximate transfer single crossing.

Definition 5 *Let P and T be posets, and let $f : P \times T \to \mathbb{R}$. The function f satisfies the upward (resp. downward) transfer ε-single-crossing property in $(p; t)$ ($\varepsilon > 0$) if for all $p' \prec p''$ (resp., $p' \succ p''$) and $t' \prec t''$ (resp., $t' \succ t''$), $f(p'', t') - f(p', t') > \varepsilon$ implies $f(\widehat{p}, t'') - f(p', t'') > \varepsilon$ for some $\widehat{p} \in P$ with $\widehat{p} \succeq p''$ (resp., $\widehat{p} \preceq p''$). The function f satisfies the approximate upward (resp., downward) transfer single-crossing property in $(p; t)$ if it satisfies the upward (resp., downward) transfer ε-single-crossing property in $(p; t)$ for every $\varepsilon > 0$.*

In Definition 5, the word 'transfer' can be omitted if $\widehat{p} = p''$. Another possible name for the approximate upward (resp., downward) single-crossing property is 'upward (resp., downward) nondecreasing positive differences,' since it can also be represented as follows: for all $p' \prec p''$ (resp., $p' \succ p''$) and $t' \prec t''$ (resp., $t' \succ t''$), $f(p'', t') - f(p', t') > 0$ implies $f(p'', t'') - f(p', t'') \geq f(p'', t') - f(p', t')$.

Approximate single-crossing allows us to study strategic complementarities expressed in terms of ε-best-reply correspondences.

Lemma 7 *Let P be a totally ordered set and T be a poset. If $f : P \times T \to \mathbb{R}$ satisfies the upward (resp., downward) transfer ε-single-crossing property in $(p; t)$ for some $\varepsilon > 0$, then the correspondence $M^\varepsilon : T \twoheadrightarrow P$ defined by $M^\varepsilon(t) = \{p \in P : f(p, t) \geq \sup_{z \in P} f(z, t) - \varepsilon\}$ is increasing downward (resp., upward).*

Proof. Let $\varepsilon > 0$, and let f satisfy the upward ε-single-crossing property in $(p; t)$. Pick some t' and t'' with $t' \prec t''$. Pick some $p'' \in M^\varepsilon(t'')$. We need to show that there exists $p' \in M^\varepsilon(t')$ such that $p' \preceq p''$. By way of contradiction, assume that it is not the case; that is, $p' \succ p''$ for every $p' \in M^\varepsilon(t')$. Then $p'' \notin M^\varepsilon(t')$; that is, $f(p'', t') < \sup_{z \in P} f(z, t') - \varepsilon$. By the definition of the least upper bound, there exists $z' \in M^\varepsilon(t')$ such that $f(z', t') - f(p'', t') > \varepsilon$. Since $z' \succ p''$ and f satisfies the upward transfer ε-single-crossing property in $(p; t)$, we have that $f(\widehat{p}, t'') - f(p'', t'') > \varepsilon$ for some $\widehat{p} \in P$ with $\widehat{p} \succeq z'$, which contradicts the fact that $p'' \in M^\varepsilon(t'')$. ■

Since increasing upward or downward correspondences need not have an increasing single-valued selection, one more condition is to be added.

Lemma 8 *Let P be a totally ordered compact space, T be a poset, and $\varepsilon \geq 0$. Let $f : P \times T \to \mathbb{R}$, and let $M^\varepsilon : T \twoheadrightarrow P$ defined by $M^\varepsilon(t) = \{p \in P : f(p, t) \geq \sup_{z \in P} f(z, t) - \varepsilon\}$ be nonempty-valued. If $f(\cdot, t) : P \to \mathbb{R}$ is upward (resp., downward) upper semicontinuous for every $t \in T$, then $\bigvee M^\varepsilon(t) \in M^\varepsilon(t)$ (resp., $\bigwedge M^\varepsilon(t) \in M^\varepsilon(t)$) for every $t \in T$.*

It is worth noticing that, in Lemma 8, the condition that the correspondence M^ε is nonempty-valued matters only when $\varepsilon = 0$. The proof of Lemma 8 is provided in the Appendix.

2.5 Better-reply security

Although, the notion of a better-reply secure game, due to Reny (1999), has been generalized in a number of ways recently, we do not need any of the generalizations for the purposes of this paper. We are interested in the property because, in the compact games, it implies that if an ε-Nash equilibrium exists for every $\varepsilon > 0$, then the game has a Nash equilibrium. However, it has turned out that studying the existence of ε-equilibria in better-reply secure games is quite challenging on its own. For compact, quasiconcave, payoff secure games, a solution in this direction was proposed by Prokopovych (2011).

We now provide some basic facts related to better-reply security, tailored for this paper's needs.

Consider a compact game $G = (X_i, u_i)_{i \in I}$, where $I = \{1, \ldots, n\}$ denotes the set of players, each strategy set X_i is a nonempty compact topological space, and each payoff function u_i is a bounded real-valued function defined on the Cartesian product $X = \prod_{i \in I} X_i$ equipped with the product topology. Denote the set of all pure strategy equilibria of G in X by E_G, and $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$. Let $\varepsilon > 0$. Player i's ε-best-reply correspondence $M_i^\varepsilon : X_{-i} \twoheadrightarrow X_i$ is defined by $M_i^\varepsilon(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) \geq \sup_{z_i \in X_i} u_i(z_i, x_{-i}) - \varepsilon\}$. A strategy profile $x = (x_1, \ldots x_n) \in X$ is an ε-Nash equilibrium of G if $x_i \in M_i^\varepsilon(x_{-i})$ for each $i \in I$. Denote the set of ε-Nash equilibria of G by $E_G(\varepsilon)$.

Better-reply security can be described as follows: A game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if and only if whenever $x \in X \backslash E_G$, there exist $\varepsilon > 0$, $d = (d_1, \ldots, d_n) \in X$, and an open neighborhood $\mathcal{N}(x)$ of x in X such that for every $y \in \mathcal{N}(x)$ there is a player i for whom $u_i(d_i, x'_{-i}) > u_i(y) + \varepsilon$ for every $x' \in \mathcal{N}(x)$ (see Prokopovych, 2013; and Reny, 2015).

Lemma 9 *Let $G = (X_i, u_i)_{i \in I}$ be a better-reply secure, compact game. Let $\{\varepsilon_k\}$ be a sequence of positive numbers converging to* 0, *and let $x_k \in E_G(\varepsilon_k)$ for* $k = 1, 2, \ldots$. *Then every cluster point of the sequence $\{x_k\}$ is a Nash equilibrium of G.*

Lemma 9, mentioned in Remark 3.1 of Reny (1999), readily follows from the above characterization of better-reply security.

3 Equilibrium existence results

This section begins with Theorem 1, an equilibrium existence result for games where each payoff function is transfer weakly upper continuous in own strategy.

Then, Theorem 2 provides a set of sufficient conditions for the existence of an equilibrium in better-reply secure games.

Let $I = \{1, \ldots, n\}$. If, for each $i \in I$, X_i is a partially ordered set with the binary relation $\preceq_i$, then $X = \prod_{i \in I} X_i$ and $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$ are posets with the corresponding product relations; that is, for example, $x \preceq y$ in X if $x_i \preceq_i y_i$ for each $i \in I$.

Definition 6 *A game $G = (X_i, u_i)_{i \in I}$ exhibits strategic complementarities if for each $i \in I$: (1) X_i is a nonempty totally ordered compact space; (2) u_i is transfer weakly upper continuous in x_i and upward or downward upper semicontinuous in x_i; (3) u_i satisfies the upward (or, resp., downward) transfer single-crossing property in $(x_i; x_{-i})$.*

That is, in the games exhibiting strategic complementarities, along with being transfer weakly upper continuous in own strategy x_i, each payoff function u_i is either upward upper semicontinuous in x_i and satisfies the upward transfer single-crossing property in $(x_i; x_{-i})$, or is downward upper semicontinuous in x_i and satisfies the downward transfer single-crossing property in $(x_i; x_{-i})$.

Theorem 1 *Every game $G = (X_i, u_i)_{i \in I}$ with strategic complementarities has a pure strategy Nash equilibrium.*

Proof. For each $i \in I$, the transfer weak upper continuity of each u_i in x_i implies that player i's best-reply correspondence $M_i : X_{-i} \twoheadrightarrow X_i$ defined by $M_i(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) = \sup_{z_i \in X_i} u_i(z_i, x_{-i})\}$ is nonempty-valued. Lemma 5 implies that each M_i is increasing upward or downward. Since each payoff function u_i is upward (or, resp., downward) upper semicontinuous in own strategy, it follows from Lemma 8 that, for every $x_{-i} \in X_{-i}$, $M_i(x_{-i})$ contains $\bigvee M_i(x_{-i})$ (or, resp., $\bigwedge M_i(x_{-i})$). Then, by Lemma 2, each M_i has an increasing selection m_i. Define an increasing function m from X to X by $m(x) = (m_1(x_{-1}), \ldots, m_n(x_{-n}))$ for $x \in X$. The set X, as the direct product of complete chains, is a complete lattice. Then, by Tarski's fixed point theorem, the function m has a fixed point. This strategy profile is a Nash equilibrium of G. $\blacksquare$

If the payoff functions of a game are not transfer upper weakly continuous in own strategies, one may attempt to use modifications of the single-crossing property designed for studying monotonicity properties of ε-best-reply correspondences, such as the above-introduced approximate upward and downward single-crossing properties.

Definition 7 *A game $G = (X_i, u_i)_{i \in I}$ exhibits approximate strategic complementarities if for each $i \in I$: (1) X_i is a nonempty totally ordered compact space; (2) u_i is bounded and upward or downward upper semicontinuous; (3) u_i satisfies the approximate upward (or, resp., downward) transfer single-crossing property in $(x_i; x_{-i})$; and, in addition, (4) G is better-reply secure.*

In particular, (1) and (2) imply that G is a compact game.

Theorem 2 *Every game $G = (X_i, u_i)_{i \in I}$ with approximate strategic complementarities has a pure strategy Nash equilibrium.*

Proof. The proof of the theorem consists of two steps. First, we show, using lattice-theoretic tools, that G has ε-Nash equilibria for every $\varepsilon > 0$, and then make use of Lemma 9, since the game is compact and better-reply secure.

Fix some $\varepsilon > 0$. It follows from Lemma 7 that each player i's ε-best-reply correspondence M_i^ε from X_{-i} to X_i is increasing downward or upward, depending on whether u_i satisfies the upward transfer ε-single-crossing property or the downward transfer ε-single-crossing property. Since u_i is either upward or, respectively, downward upper semicontinuous, Lemmas 2 and 8 imply that each M_i^ε has an increasing selection m_i^ε from X_{-i} to X_i. Then the function $m^\varepsilon : X \to X$ defined by $m^\varepsilon(x) = (m_1^\varepsilon(x_{-1}), \ldots, m_n^\varepsilon(x_{-n}))$ for $x \in X$ is increasing. By Tarski's fixed point theorem, it has a fixed point.

Therefore, G has an ε-Nash equilibrium for every $\varepsilon > 0$. By Lemma 9, G has a pure strategy Nash equilibrium. ■

4 Applications

This section explains, with the aid of economics-related examples, the paper's major contributions. The partnership game studied in Example 4 illustrates the strengths of the generalized upper semicontinuity conditions used in Theorem 1. In the game, the players' payoff functions are not upper semicontinuous in own strategies, and, moreover, their best-reply correspondences are neither Veinott-increasing upward nor Veinott-increasing downward.

In order to apply Theorem 1 to the war of attrition game studied in Example 5, it is enough to reverse the natural order on player 2's strategy set. Then, player 1's payoff function satisfies the upward single-crossing property in $(x_1; x_2)$ and player 2's payoff function satisfies the downward single-crossing property in $(x_2; x_1)$. Consequently, player 1's best-reply correspondence is Veinott-increasing upward, and player 2's is Veinott-increasing downward.

Example 6 is a Bertrand duopoly model with homogeneous products. Reny's (1999) equilibrium existence theorem can not be applied to the game because it is not quasiconcave. Vives's (1990) and Milgrom and Shannon's (1994) results can not be applied to it because the players' payoff functions are too discontinuous in own strategies. The existence of a Nash equilibrium in Example 6 follows from Theorem 2, where the two mentioned approaches are integrated.

Example 4 Each of two partners has no more than one unit of effort to contribute to a project. If each partner i chooses the amount of effort e_i, the total output is $f(e_1, e_2) = e_1 + e_2$. Given a profile (e_1, e_2), partner i obtains the fraction $p_i(e_i, e_{-i})$ of the total output, where

$$p_i(e_i, e_{-i}) = \begin{cases} 1 \text{ if } e_i > e_{-i} \\ \frac{1}{2} \text{ if } e_1 = e_2 \\ 0 \text{ if } e_i < e_{-i}. \end{cases}$$

In this game, player i's payoff function $u_i : [0,1] \times [0,1] \to \mathbb{R}$ is defined by $u_i(e_i, e_{-i}) = p_i(e_i, e_{-i})(e_i + e_{-i}) - e_i$. Player i's best-reply correspondence $M_i : [0,1] \twoheadrightarrow \mathbb{R}$ is the following:

$$M_i(e_{-i}) = \begin{cases} [0,1] \text{ if } e_{-i} = 0 \\ (e_{-i}, 1] \text{ if } e_{-i} \in (0,1) \\ \{0,1\} \text{ if } e_{-i} = 1. \end{cases}$$

Assuming that the players' strategy sets are equipped with the natural order, let us, for example, look, in some detail, at the properties of the correspondence M_i. It is neither Veinott-increasing upward ($\frac{1}{2} \in M_i(0)$, $0 \in M_i(1)$, but $\frac{1}{2} \notin M_i(1)$) nor increasing downward ($1 \in M_i(\frac{1}{2})$, $0 \in M_i(1)$, but $0 \notin M_i(\frac{1}{2})$). However, M_i is increasing upward since $1 \in M_i(e_{-i})$ for every $e_{-i} \in [0,1]$. One can also verify that each u_i satisfies the upward transfer single crossing property in (e_i, e_{-i}), but not the upward single crossing property ($u_i(\frac{1}{2}, \frac{1}{4}) - u_i(0, \frac{1}{4}) = \frac{1}{4}$, but $u_i(\frac{1}{2}, 1) - u_i(0,1) = -\frac{1}{2}$). Since, in e_i, each u_i is transfer weakly upper semicontinuous and upward upper semicontinuous, it follows from Theorem 1 that the game has a pure strategy Nash equilibrium.

Example 5 Consider the following war of attrition game G. Two players compete for an object over the time interval $[0, c]$. The players' valuations of the object are equal to v_1 and v_2, where $0 < v_2 \leq v_1 < c$. Player i's set of strategies T_i is the set of possible concession times, $[0, c]$. Player i's payoff function is as follows:

$$u_i(t_i, t_{-i}) = \begin{cases} -t_i \text{ if } t_i < t_{-i}, \\ \frac{1}{2}v_i - t_i \text{ if } t_i = t_{-i}, \\ v_i - t_{-i} \text{ if } t_i > t_{-i}. \end{cases}$$

Player i's best-reply correspondence $M_i : T_{-i} \twoheadrightarrow T_i$ is given by:

$$M_i(e_{-i}) = \begin{cases} (t_{-i}, c] \text{ if } t_{-i} < v_i \\ \{0\} \cup (t_{-i}, c] \text{ if } t_{-i} = v_i \\ \{0\} \text{ if } t_{-i} > v_i. \end{cases}$$

Clearly, each M_i is neither increasing upward nor increasing downward.

Consider the game G^- where the payoff functions are the same as those in G, but the order on player 2's strategy set is reversed. Thus, in G^-, $t_1'' \succeq_1 t_1'$ if and only if $t_1'' \geq t_1'$ for every t_1', $t_1'' \in [0, c]$, and $t_2'' \succeq_2 t_2'$ if and only if $t_2'' \leq t_2'$ for every t_2', $t_2'' \in [0, c]$.

It is not difficult to see that, in G^-, u_1 is upward upper semicontinuous in t_1 and satisfies the upward single-crossing property in $(t_1; t_2)$. To check the latter, pick some $t_2'' < t_2'$ and $t_1' < t_1''$ such that $u_1(t_1'', t_2') - u_1(t_1', t_2') \geq 0$. It is worth noticing that the last inequality implies that $t_1'' \geq t_2'$. We need to show that $u_1(t_1'', t_2'') - u_1(t_1', t_2'') \geq 0$. If $t_2'' \leq t_1'$, then $u_1(t_1'', t_2'') - u_1(t_1', t_2'') \geq (v_1 - t_2'') - (v_1 - t_2'') = 0$. If $t_1' < t_2''$, then $u_1(t_1', t_2') = u_1(t_1', t_2'') = -t_1'$ and, since $t_1'' \geq t_2' > t_2''$, $u_1(t_1'', t_2'') > u_1(t_1'', t_2')$. Consequently, $u_1(t_1'', t_2'') - u_1(t_1', t_2'') > u_1(t_1'', t_2') - u_1(t_1', t_2')$.

Similarly, u_2 is downward upper semicontinuous in t_2 and satisfies the downward single-crossing property in $(t_2; t_1)$. Therefore, in G^-, player 1's best-reply correspondence is Veinott-increasing upward and player 2's best-reply correspondence is Veinott-increasing downward. Thus, the existence of a Nash equilibrium in the game G^- follows from Theorem 1.

In the next example, we use Theorem 2 to show that a nonquasiconcave Bertrand duopoly model with a nonlinear aggregate demand curve has a pure strategy Nash equilibrium. It is useful to notice that Reny's equilibrium existence theorem also implies that the game has a mixed strategy Nash equilibrium.[5]

Example 6 Consider the following Bertrand duopoly model with homogeneous products. There are two identical firms with the total cost functions $C_i(q_i) = q_i$, $i = 1, 2$. The demand function $D : [0, +\infty) \to [0, +\infty)$ is as follows:

$$D(p) = \begin{cases} 20 - 3p \text{ if } p \in [0, 4], \\ 10 - \frac{1}{2}p \text{ if } p \in [4, 20], \\ 0 \text{ if } p \in (20, +\infty). \end{cases}$$

Its graph is not a straight line, but has a conventional convex shape.

[5] See, e.g., Monteiro and Page (2007), Allison and Lepore (2014), Prokopovych and Yannelis (2014).

Then, firm i's profit function $u_i : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is given by

$$u_i(p_i, p_{-i}) = \begin{cases} (p_i - 1)D(p_i) \text{ if } p_i < p_{-i}, \\ \frac{1}{2}(p_i - 1)D(p_i) \text{ if } p_i = p_{-i}, \\ 0 \text{ if } p_i > p_{-i}. \end{cases}$$

There is no loss of generality in assuming that each player i's strategy set is $X_i = [1, 20]$ because, for each player i, every strategy from the set $[0, 1) \cup (20, +\infty)$ is weakly dominated by, for example, $p_i = 2$. Consequently, if the game has a pure strategy Nash equilibrium in $[1, 20] \times [1, 20]$, then the strategy profile is also a Nash equilibrium of the entire game.

It is useful to notice that the maximizer of the function $f_1 : [1, 4] \to \mathbb{R}$ defined by $f_1(p) = (p - 1)(20 - 3p)$ is $p_1 = \frac{23}{6}$ and the maximizer of the function $f_2 : [4, 20] \to \mathbb{R}$ defined by $f_2(p) = (p - 1)(10 - \frac{1}{2}p)$ is $p_2 = 10.5$. Also notice that $f_1(p) = f_2(p)$ at $p = 4$. Since, for example, the set of player 1's profitable deviations from the strategy profile $(p_1, p_2) = (4, 11)$ contains both $\frac{23}{6}$ and 10.5, the game is not quasiconcave. The payoff functions are not transfer weakly upper continuous in own strategies. Consequently, some values of the best-reply correspondences are empty. Verifying whether the ε-best-reply correspondences are increasing downward is reduced, in virtue of Lemma 7, to verifying whether each player i's payoff function satisfies the approximate upward transfer single-crossing property in $(p_i; p_{-i})$. We now show this for player 1's payoff function.

Fix some $\varepsilon > 0$. Consider some p_1' and p_1'' in $[1, 20]$ with $p_1' < p_1''$ and some p_2' and p_2'' in $[1, 20]$ with $p_2' < p_2''$ such that $u_1(p_1'', p_2') - u_1(p_1', p_2') > \varepsilon$. We want to show that $u_1(p_1'', p_2'') - u_1(p_1', p_2'') > \varepsilon$.

Notice that p_2' can not be less than p_1''; otherwise, the difference $u_1(p_1'', p_2') - u_1(p_1', p_2')$ would be nonpositive. Then $p_2'' > p_2' \geq p_1'' > p_1'$, and, therefore, $u_1(p_1', p_2') = u_1(p_1', p_2'')$ and $u_1(p_1'', p_2') \leq u_1(p_1'', p_2'')$. Thus, $u_1(p_1'', p_2'') - u_1(p_1', p_2'') \geq u_1(p_1'', p_2') - u_1(p_1', p_2') > \varepsilon$.

It is not difficult to see that each payoff function is downward upper semicontinuous in own strategy. Since the game is better-reply secure, it has a pure strategy Nash equilibrium by Theorem 2.

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5 Conclusions

Lattice-theoretic tools can be used to study equilibrium existence in strategic games with totally ordered strategy sets where the payoff functions are not upper semicontinuous in own strategies and do not satisfy the single-crossing property. If the payoff functions are transfer weakly upper continuous in own strategies, the existence of a Nash equilibrium follows from directional upper semicontinuity and directional transfer single crossing. In games where best-reply correspondences are not necessarily nonempty-valued everywhere, strategic complementarities might reveal themselves in the ε-best-reply correspondences. If so, directional approximate transfer single crossing is employed, along with better-reply security and directional upper semicontinuity. The major results of this paper are illustrated with the aid of a number of economics-related examples to which the seminal contributions by Vives (1990), Milgrom and Shannon (1994), and Reny (1999) are not applicable.

Appendix

The Appendix contains proofs of some auxiliary lemmas.

Proof of Lemma 1

Assume first that f is order upper semicontinuous and a net $\{p_\alpha\}$ converges to $\widehat{p}$ in P. We need to show that $\limsup_\alpha f(p_\alpha) \leq f(\widehat{p})$. By passing to a subnet if necessary, we may assume, without loss of generality, that $\limsup_\alpha f(p_\alpha) = \lim_\alpha f(p_\alpha)$. Since $\{p_\alpha\}$ is totally ordered, it has a monotone (increasing or decreasing) subnet, again denoted by $\{p_\alpha\}$, converging to $\widehat{p}$ (see, e.g., Roman, 2008, p. 17). It follows from the order upper semicontinuity of f that $f(\widehat{p}) \geq \lim_\alpha f(p_\alpha)$.

Conversely, let f be upper semicontinuous, and let $\{p_\alpha\}$ be, for example, an increasing net of elements of P. Since P is compact, $\{p_\alpha\}$ converges to $\widehat{p} = \bigvee_\alpha p_\alpha$. Then $\limsup_\alpha f(p_\alpha) \leq f(\bigvee_\alpha p_\alpha)$ by the upper semicontinuity of f. A similar reasoning can be provided for a decreasing net.

Proof of Lemma 8

Consider, for example, the case when f is downward upper semicontinuous on P for every $t \in T$. Fix some $t \in T$. We need to show that there exists $\widehat{p} \in M^\varepsilon(t)$ such that $\widehat{p} \preceq p$ for every $p \in M^\varepsilon(t)$. Since P is compact, the closure of $M^\varepsilon(t)$, $\mathrm{cl}M^\varepsilon(t)$, is also compact. Since $\mathrm{cl}M^\varepsilon(t)$ is also a chain, it has a least element $\widehat{p} \in \mathrm{cl}M^\varepsilon(t)$

such that $\widehat{p} \preceq p$ for every $p \in \mathrm{cl}M^{\varepsilon}(t)$. Then $\widehat{p}$ is the limit of a net $\{p_\alpha\}$ in $M^{\varepsilon}(t)$. Since P is totally ordered, $\{p_\alpha\}$ has a decreasing subnet that converges to $\widehat{p}$. Denote it again $\{p_\alpha\}$. Since $f(\cdot, t) : P \to \mathbb{R}$ is downward upper semicontinuous and $\widehat{p} = \bigwedge_{\alpha} p_\alpha$, $\limsup_\alpha f(p_\alpha, t) \le f(\widehat{p}, t)$. Therefore, $\widehat{p} \in M^{\varepsilon}(t)$; that is, $\widehat{p}$ is not only the least element of $\mathrm{cl}M^{\varepsilon}(t)$, but is also the least element of $M^{\varepsilon}(t)$.

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